Relationship of Poisson and exponential distributions

KJC (02/15/99)

The question we are trying to answer is, what is the distribution of the time between events in a Poisson process? Recall that the probability function for the Poisson distribution is,

$$f(x) = \Pr[X = x] = \left(\frac{(\lambda t)^x}{x!}\right)e^{-\lambda t} \tag{1}$$

where, λ is the mean rate of arrivals and t is a period of time. Defining T as the time of an event (a random variable), we have (by definition),

$$F(t) = \Pr[T \le t]. \tag{2}$$

This is equal to,

$$F(t) = \Pr[T \le t] = 1 - \Pr[T > t] \tag{3}$$

where (and this is the key step so pay attention here ☺),

$$\Pr[T > t] = \Pr[\text{zero events occur in time 0 to t}] \;=\; \Pr[\mathrm{X} = 0] \;=\; \left(\frac{(\lambda \mathrm{t})^0}{0!}\right)e^{-\lambda t} = e^{-\lambda t} \tag{4}$$

Now, we plug in our result for $\Pr[T > t]$ into our equation (3) and we get,

$$F(t) = 1 - e^{-\lambda t} \tag{5}$$

which is the distribution function for the exponential distribution (check your notes!).

Memoryless property of the exponential distribution

Memoryless means that the expected time until the next event is the same no matter how long since the last event occurred. That is,

$$\Pr(x \le T + t \mid x > T) = \Pr(x \le t) \tag{6}$$

This seems to be not "real world". For example, the longer a real traffic light has been red, the greater the probability that it will turn green in the next, say, 10 seconds. If the probability of a traffic light turning green in the next 10 seconds does not change independent of how long it has been red, then the distribution of the red light is memoryless. Only two distributions are memoryless - the exponential (continuous) and geometric (discrete). Here is the memoryless proof for the exponential distribution…

$$\Pr(x \le T + t \mid x > T) = \frac{\Pr\left[(x \le T + t) \cap (x > T)\right]}{\Pr(x > T)} = \frac{\Pr(x \le T + t) - \Pr(x \le T)}{\Pr(x > T)}$$

$$= \frac{\left(1 - e^{-\lambda(T+t)}\right) - \left(1 - e^{-\lambda T}\right)}{1 - \left(1 - e^{-\lambda T}\right)} = \frac{e^{-\lambda T}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda T}} = 1 - e^{-\lambda t} \qquad \text{QED ☺} \tag{7}$$