Summary of Key Probability Distributions

This handout contains a summary of some important probability distributions for performance modeling of computer systems and communications networks. The distributions summarized here are uniform (continuous), uniform (discrete), binomial, Poisson, exponential, hyperexponential-2, Pareto, and bounded Pareto. Reference is made to tools on the Christensen tools page (http://www.csee.usf.edu/~christen/tools/toolpage.html) and to the Mesquite Software CSIM product.

**Uniform distribution (continuous):**

A random variable with uniform distribution in \( a \leq x \leq b \) has probability density function,

\[
f(x) = \begin{cases} 
\frac{1}{b-a} & a \leq x \leq b \\
0 & \text{otherwise} 
\end{cases}
\]

and a probability distribution function,

\[
F(x) = \begin{cases} 
0 & x < a \\
\frac{(x-a)}{(b-a)} & a \leq x < b \\
1 & x \geq b 
\end{cases}
\]

The mean and variance are,

\[
\mu = \frac{1}{2}(a + b) \quad \text{and} \quad \sigma^2 = \frac{1}{12}(b-a)^2.
\]

To generate use genunifc.c from Christensen tools page or uniform() in the CSIM library.

**Uniform distribution (discrete):**

A random variable with uniform distribution in \( a \leq k \leq b \) where \( n = b - a + 1 \) has a probability mass function,

\[
f(k) = \begin{cases} 
1/n & a \leq k \leq b \\
0 & \text{otherwise} 
\end{cases}
\]

and a cumulative distribution function,

\[
F(k) = \begin{cases} 
0 & k < a \\
(k-a)/n & a \leq k \leq b \\
1 & k > b 
\end{cases}
\]

The mean and variance are,

\[
\mu = \frac{1}{2}(a + b) \quad \text{and} \quad \sigma^2 = \frac{1}{12}(n^2 - 1).
\]

To generate use genunifd.c from Christensen tools page or random_int() in the CSIM library.
**Binomial distribution (discrete):**

A random variable with binomial distribution for \( n \) trials with probability \( p \) of success for each trial has probability mass function,

\[
f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n
\]

The cumulative distribution function is messy.

The mean and variance are,

\[
\mu = np \quad \text{and} \quad \sigma^2 = np(1-p).
\]

**Note:** Define \( \lambda = np \), then as \( n \) goes to infinity the binomial distribution tends to the Poisson distribution with rate \( \lambda \).

To generate use `genbin.c` from Christensen tools page or `binomial()` in the CSIM library.

**Poisson distribution (discrete):**

A random variable with Poisson distribution for a rate \( \lambda \) of arrivals has probability mass function,

\[
f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, ...
\]

The cumulative distribution function is messy.

The mean and variance are,

\[
\mu = \lambda \quad \text{and} \quad \sigma^2 = \lambda.
\]

**Note:** The distribution of time between arrivals in Poisson process is exponentially distributed with mean \( 1/\lambda \).

To generate use `genpois.c` from Christensen tools page or `poisson()` in the CSIM library.

**Exponential distribution (continuous):**

A random variable with exponential distribution has density function,

\[
f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}
\]

and distribution function,

\[
F(t) = \begin{cases} 1-e^{-\lambda t} & x > 0 \\ 0 & x \leq 0 \end{cases}
\]

The mean and variance are,

\[
\mu = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{1}{\lambda^2}.
\]

To generate use `genexp.c` from Christensen tools page or `exponential()` in the CSIM library.
Hyperexponential-2 distribution (continuous):

A random variable with hyperexponential-2 distribution with parameters $\lambda_1$, $\lambda_2$, and $p$ has density function,

$$f(t) = \begin{cases} p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

and distribution function,

$$F(t) = \begin{cases} 1 - pe^{-\lambda_1 t} - (1-p)e^{-\lambda_2 t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

The mean and variance are,

$$\mu = \frac{p}{\lambda_1} + 1 - p \lambda_2\lambda_1^{-1} \quad \text{and} \quad \sigma^2 = 2\left(\frac{p}{\lambda_1^2} + 1 - p\lambda_2\lambda_1^{-1}\right) - \left(\frac{p}{\lambda_1} + 1 - p\lambda_2\lambda_1^{-1}\right)^2.$$  

To generate use genhyp1.c or genhyp2.c from Christensen tools page or hyperx() in the CSIM library.

Pareto distribution (continuous):

A random variable with Pareto distribution with shape parameter $\alpha$ and minimum value $k$ has density function,

$$f(x) = \begin{cases} \frac{\alpha k^\alpha}{x^{\alpha+1}} & x \geq k \\ 0 & x < k \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 1 - \left(\frac{k}{x}\right)^\alpha & x \geq k \\ 0 & x < k \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha k}{\alpha - 1} \quad \text{and} \quad \sigma^2 = \frac{\alpha k^2}{(\alpha - 1)(\alpha - 2)}.$$  

Note: The Pareto distribution is heavy tailed. The mean is infinity for $\alpha < 1$ and the variance is infinity for $\alpha < 2$.

To generate use genpar1.c from Christensen tools page or pareto() in the CSIM library.
Bounded Pareto distribution (continuous):

A random variable $X$ with Bounded Pareto distribution with shape parameter $\alpha$, minimum value $k$, and maximum value $p$ has density function,

$$f(x) = \begin{cases} 0 & x < k \\ \frac{\alpha \cdot k^\alpha}{x^{-\alpha - 1}} & k \leq x \leq p \\ \frac{1 - \left( \frac{k}{p} \right)^\alpha}{x^{-\alpha} - p^\alpha} & x > p \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 0 & x < k \\ \frac{p^\alpha \left( k^\alpha - x^\alpha \right)}{x^\alpha \left( k^\alpha - p^\alpha \right)} & k \leq x \leq p \\ 0 & x > p \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha \cdot k^\alpha \cdot p^{1-\alpha} - k}{(\alpha - 1) \left( \frac{k}{p} \right)^\alpha - 1} \quad \text{and} \quad \sigma^2 = \frac{\alpha \cdot k^\alpha \cdot p^{2-\alpha} - k^2}{(\alpha - 2) \left( \frac{k}{p} \right)^\alpha - 1} - \frac{\alpha^2 \cdot (k - k^\alpha \cdot p^{1-\alpha})^2}{(\alpha - 1)^2 \left( \frac{k}{p} \right)^\alpha - 1}.$$ 

**Note:** The Bounded Pareto distribution is effectively heavy tailed, but has finite mean and variance.

To generate use `genpar2.c` from Christensen tools page (there is no CSIM library function to generate Bounded Pareto random variables).

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