# Complexity Results in Graph Reconstruction*

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**Abstract.** We investigate the relative complexity of the graph isomorphism problem (GI) and problems related to the reconstruction of a graph from its vertex-deleted or edge-deleted subgraphs. We show that the problems are rather closely related for all amounts $c$ of deletion:

1. For all $c \geq 1$, GI $\equiv^l_{iso}$ VDC$_c$, GI $\equiv^l_{iso}$ EDC$_c$, GI $\leq^l_m$ LVD$_c$, and GI $\equiv^p_{iso}$ LED$_c$.
2. For all $c \geq 1$ and $k \geq 2$, GI $\equiv^p_{iso}$ $k$-VDC$_c$ and GI $\equiv^p_{iso}$ $k$-EDC$_c$.
3. For all $c \geq 1$ and $k \geq 2$, GI $\leq^l_m$ $k$-LVD$_c$. In particular, for all $c \geq 1$, GI $\equiv^p_{iso}$ 2-LVD$_c$.
4. For all $c \geq 1$ and $k \geq 2$, GI $\equiv^p_{iso}$ $k$-LED$_c$.

For many of these, even the $c = 1$ cases were not known.

Similar to the definition of reconstruction numbers $vrn_\exists(G)$ [HP85] and $ern_\exists(G)$ (see p. 120 of [LS03]), we introduce two new graph parameters, $vrn_\forall(G)$ and $ern_\forall(G)$, and give an example of a family $\{G_n\}_{n \geq 4}$ of graphs on $n$ vertices for which $vrn_\exists(G_n) < vrn_\forall(G_n)$. For every $k \geq 2$ and $n \geq 1$, we show there exists a collection of $k$ graphs on $(2^{k-1}+1)n+k$ vertices with $2^n$ 1-vertex-preimages, i.e., one has families of graph collections whose number of 1-vertex-preimages is huge relative to the size of the graphs involved.

## 1 Introduction

### 1.1 Background

The general form of a combinatorial reconstruction problem is the following: Given a mathematical structure $\mathcal{S}$ (e.g., a graph, a hypergraph, a characteristic polynomial of a graph, etc.) and a collection $\mathcal{D}(\mathcal{S})$ of its associated substructures (e.g., vertex-deleted subgraphs, edge-deleted subgraphs, characteristic polynomial of subgraphs, etc.), is it possible to reconstruct $\mathcal{S}$ from $\mathcal{D}(\mathcal{S})$ with some minor or no imperfections? This reconstruction problem is interesting not only from the mathematical point of view but also for its diverse applicability in several

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fields. In bioinformatics, the multiple sequence alignment problem (MSA) [CL88] is to reconstruct a sequence with minimum gap insertion and maximum number of matching symbols, given a list of protein or DNA sequences. In computer networking, the reconstruction problem appears in the following scenario: Given a collection of sketches depicting partial network connections in a city from different locations, construct the network connection in the entire city.

In this paper, we are concerned with reconstruction problems arising in graph theory. The foremost open problems in the theory of reconstruction of graphs are the Reconstruction Conjecture and the Edge-Reconstruction Conjecture. The Reconstruction Conjecture, formulated by Kelly and Ulam in 1942 [Kel42,Ula60], asserts that every finite simple undirected graph on at least three vertices is determined uniquely (up to isomorphism—we treat our graphs broadly as unlabeled) by its collection of 1-vertex-deleted subgraphs. Harary [Har64] formulated the Edge-Reconstruction Conjecture, which states that a finite simple graph with at least four edges can be reconstructed from its collection of 1-edge-deleted subgraphs. For more on these conjectures, the reader can refer to a number of survey papers (see, for instance, [BH77,NW78,Man88,Bon91]) and the book [LS03].

Nash-Williams [NW78] posed an interesting computational problem: Given a collection of graphs, how can we decide whether this has been generated from some graph by deleting one vertex every possible way, i.e., whether the collection is *legitimate*? A similar problem has been posed where we ask whether the collection is generated from some graph by deleting one edge every possible way. These problems are known as the *Legitimate Vertex-Deck Problem* (LVD) and the *Legitimate Edge-Deck Problem* (LED), respectively. Other, seemingly easier, problems are the *Vertex-Deck Checking Problem* (VDC) and the *Edge-Deck Checking Problem* (EDC) where, given a graph $G$ and a collection $\mathcal{D}$ of graphs, we ask whether $\mathcal{D}$ can be generated from $G$ by deleting one vertex, respectively one edge, every possible way.

Mansfield [Man82] and Kratsch and Hemaspaandra [KH94] studied complexity aspects of legitimate deck problems and deck checking problems. Kratsch and Hemaspaandra [KH94] showed that LVD is logspace many-one hard for the Graph Isomorphism problem (this extends the earlier result of Mansfield [Man82], of which [KH94] were unaware, that polynomial-time many-one hardness holds). Mansfield [Man82] showed that LED is polynomial-time Turing equivalent to the Graph Isomorphism problem (GI). Kratsch and Hemaspaandra [KH94] proved that GI is logspace isomorphic to VDC and obtained polynomial-time algorithms for LVD when restricted to certain classes of graphs—including graphs of bounded degree, partial $k$-trees for any fixed $k$, and graphs of bounded genus. Köbler, Schöning, and Torán [KST93] showed that if the Reconstruction Conjecture holds then LVD is in the class LWPP [FFK94]. Thus, conditional on the truth of the Reconstruction Conjecture, they showed that LVD is low for PP, i.e., $\mathrm{PP}^{\mathrm{LVD}} = \mathrm{PP}$. This result can be viewed as suggesting that LVD cannot be NP-complete, since if it were NP-complete then the result of [KST93] would immediately imply that either the Reconstruction Conjecture fails or $\mathrm{PP}^{\mathrm{NP}} = \mathrm{PP}$. But both these claims are widely suspected to be false.

### 1.2 Our Contributions

A more general reconstruction problem deals with collections consisting of all subgraphs obtained through the deletion of (exactly) some fixed number $c \geq 1$ of vertices (or edges). Kelly [Kel57] first raised the possibility of deleting several vertices from a graph, Manvel [Man74] made some observations on this problem, and Bondy (see Section 11.2 in [Bon91]) surveyed results on this more general reconstruction problem. (Also, see a review [Nýd01] on the progress made on this problem in the past three decades.) In this paper, one of our investigations is of the complexity of legitimate deck problems and deck checking problems for the general case when these problems are defined in terms of deletion of some fixed number $c \geq 1$ of vertices (or edges) of a graph. We observe that the logspace isomorphism between GI and VDC [KH94] holds, for every $c \geq 1$, between GI and $\mathrm{VDC}_c$ and between GI and $\mathrm{EDC}_c$. (Here and henceforth, the subscript "$c$" in the name of a problem refers to the more general problem based on the deletion of $c$ vertices or edges of a graph.) We strengthen the result of [Man82] to show that, for every $c \geq 1$, GI is, in fact, polynomial-time isomorphic to $\mathrm{LED}_c$. For $\mathrm{LVD}_c$, we observe that for every $c \geq 1$, GI $\leq_m^p$ $\mathrm{LVD}_c$ (the $c = 1$ case of this already follows from [KH94]).

We next look at the question of reconstructing a graph from a subdeck (a subset of all possible vertex-deleted or edge-deleted subgraphs). See [HP66,Bon69, Lau83] for this line of investigation in the reconstruction of trees. Our results on the complexity aspects of the reconstruction of a graph from a subdeck are described in Section 3.2. Again, we obtain a strong relationship between these problems and the graph isomorphism problem. Harary and Plantholt [HP85] introduced a parameter, called the ally-reconstruction number $vrn_{\exists}(G)$ of a graph $G$, and defined it as the minimum number of 1-vertex-deleted subgraphs needed to identify $G$ (as always, up to isomorphism). A similar definition is used for the reconstruction number $ern_{\exists}(G)$, which is defined in terms of 1-edge-deleted subgraphs (see p. 120 of [LS03]). We introduce two new parameters, $vrn_{\forall}(G)$ and $ern_{\forall}(G)$, for a graph $G$ and give an example of a family $\{G_n\}_{n\geq 4}$ of graphs on $n$ vertices for which $vrn_{\exists}(G_n) < vrn_{\forall}(G_n)$. We also give a family of collections of $k$ graphs on $(2^{k-1}+1)n+k$ vertices with $2^n$ 1-vertex-preimages, thus constructing an exponential richness of number of preimages. *(Due to space limitations, in this version most proofs are omitted; please refer to the (in-preparation) full version.)*

## 2 Preliminaries

### 2.1 Notation

Our alphabet is $\Sigma = \{0, 1\}$. Let $[., \ldots, .]$ denote multisets. We use $\cup$ to denote set union as well as multiset union. Let $\langle \ldots \rangle$ be a multi-arity, polynomial-time computable and polynomial-time invertible pairing function. We tacitly assume that multisets and graphs are encoded in a standard fashion. For background in complexity theory and for notions such as P, NP, reductions and completeness, we refer the reader to any textbook on complexity theory [HO02]. We consider only finite, undirected graphs with no self-loops. Given a graph $G$, let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of $G$. For notational convenience, we alternatively represent a graph $G$ by $(V, E)$ where $V = V(G)$ and $E = E(G)$. The degree of a vertex $v$ in $G$, denoted by $deg_G(v)$, is the number of edges incident on $v$. $\delta(G) = \min\{deg_G(v) \mid v \in V(G)\}$ and $\lambda(G)$ is the minimum number of edges whose deletion from $G$ disconnects $G$. The neighborhood $N_G(v)$ of a vertex $v$ in a graph $G$ is the set of vertices that are at a distance at most one from $v$, that is, $N_G(v) = \{v\} \cup \{w \mid \{v, w\} \in E(G)\}$. The union of graphs $G_1, G_2, \ldots, G_k$, $k \geq 2$, is denoted by $G = G_1 \cup G_2 \cup \cdots \cup G_k$ where $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and $E(G) = \bigcup_{i=1}^{k} E(G_i)$. For a graph $G$, and an integer $m > 0$, $mG$ represents the union of $m$ vertex-disjoint (isomorphic) copies of $G$. The join of graphs $G_1, \ldots, G_k$, $k \geq 2$, is denoted by $G = G_1 + \ldots + G_k$ where $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and $E(G) = \bigcup_{i=1}^{k} E(G_i) \cup \bigcup_{i \neq j}\{\{u, v\} \mid u \in V(G_i) \wedge v \in V(G_j)\}$. The notions of union and join of graphs will always require disjoint sets of vertices and thus for graphs $G$ and $H$ with $V(G) \cap V(H) \neq \emptyset$, we assume that isomorphs $\hat{G}$ and $\hat{H}$ of $G$ and $H$, respectively, with $V(\hat{G}) \cap V(\hat{H}) = \emptyset$, are used in place of $G$ and $H$. For $n \geq 1$, $K_n$ is the complete graph and $P_n$ is the path graph on $n$ vertices. That is, $(V(P_n), E(P_n)) = (\{1, \ldots, n\}, \{\{i, i+1\} \mid 1 \leq i \leq n-1\})$. The line graph $L(G)$ of a graph $G$ is defined by: $V(L(G)) = E(G)$ and $E(L(G)) = \{\{e_1, e_2\} \mid e_1, e_2 \in E(G) \wedge e_1$ and $e_2$ have exactly one vertex in common$\}$.

Given a graph $G$ and a set $S \subseteq V(G)$, $G-S$ denotes a graph with $V(G-S) = V(G) - S$ and $E(G-S) = E(G) - \{\{u, v\} \mid \{u, v\} \in E(G) \wedge \{u, v\} \cap S \neq \emptyset\}$. Similarly, if $S \subseteq E(G)$, then $G - S$ denotes a graph with $V(G-S) = V(G)$ and $E(G-S) = E(G) - S$. We will call any collection of graphs with an identical number of vertices a "vertex-deck" and will use the term "edge-deck" to denote a collection of graphs with identical numbers of edges. The graphs in a vertex-deck are called vertex-cards and the graphs in an edge-deck are called edge-cards. For a graph $G$ and for any $c \geq 1$, the $c$-vertex-deleted-deck of $G$, denoted by vertex-deck$_c(G)$, is the multiset $[G - S \mid S \subseteq V(G)$ and $||S|| = c]$, and the $c$-edge-deleted-deck of $G$, denoted by edge-deck$_c(G)$, is the multiset $[G - S \mid S \subseteq E(G)$ and $||S|| = c]$. We say that a vertex-deck $D_1 = [G_1, \ldots, G_n]$ is equivalent to a vertex-deck $D_2 = [G'_1, \ldots, G'_n]$, denoted by $D_1 = D_2$, if there exists a one-one mapping that maps each graph from $D_1$ to an isomorphic graph from $D_2$. We use a similar definition for the equivalence of two edge-decks: An edge-deck $D_1 = [G_1, \ldots, G_m]$ is equivalent to an edge-deck $D_2 = [G'_1, \ldots, G'_m]$, denoted by $D_1 = D_2$, if there exists a one-one mapping that maps each graph from $D_1$ to an isomorphic graph from $D_2$. The notion of $\mathcal{D}_1 \subseteq \mathcal{D}_2$ is defined analogously. For any $c \geq 1$, we say a graph $G$ is a $c$-vertex-preimage of $[G_1, \ldots, G_k]$ if $[G_1, \ldots, G_k] \subseteq$ vertex-deck$_c(G)$, and say a graph $G$ is a $c$-edge-preimage of $[G_1, \ldots, G_k]$ if $[G_1, \ldots, G_k] \subseteq$ edge-deck$_c(G)$. For any $c \geq 1$, we say that a graph $H$ is a $c$-vertex-card (respectively, $c$-edge-card) of a graph

$G$ if $H \in$ vertex-deck$_c(G)$ (respectively, $H \in$ edge-deck$_c(G)$). $[G_1, \ldots, G_k]$ is a legitimate $c$-vertex-deck ($c$-vertex-subdeck) if there is a graph $G$ such that $[G_1, \ldots, G_k] =$ vertex-deck$_c(G)$ ($[G_1, \ldots, G_k] \subseteq$ vertex-deck$_c(G)$). The notions of legitimate $c$-edge-deck and legitimate $c$-edge-subdeck, for any $c \geq 1$, are defined in a similar way. For any graph $G$, the endvertex-deck of $G$, denoted by endvertex-deck$(G)$, is the multiset consisting of the subgraphs $G - v$ where $v$ is an endvertex of $G$, i.e., $deg_G(v) = 1$.

### 2.2 Graph Isomorphism

A graph $G$ is isomorphic to a graph $H$ if there is a bijective mapping $\psi : V(G) \to V(H)$ such that, for all $v_1, v_2 \in V(G)$, $\{v_1, v_2\} \in E(G)$ if and only if $\{\psi(v_1), \psi(v_2)\} \in E(H)$. In this case, $\psi$ is called an isomorphism between graphs $G$ and $H$, and we write $G \cong H$. If $G \cong H$ via the identity mapping, then we use $G = H$ to represent this fact. A yes-instance $\langle G, H\rangle$ of the graph isomorphism problem (GI) is an encoding of graphs $G$ and $H$ where $G$ is isomorphic to $H$.

**Definition 1 ([KST93], see also [Kad88,RR92,LT92]).** *An* ***or****-function for a set $A$ is a function $f$ mapping sequences of strings to strings such that for every sequence $x_1, \ldots, x_n$, it holds that $f(\langle x_1, \ldots, x_n\rangle) \in A \iff (\exists i \in \{1, \ldots, n\})[x_i \in A]$. An* ***and****-function for a set $A$ can be defined similarly.*

**Proposition 1 ([KST93]).** GI *has a polynomial-time computable* ***or****-function and a polynomial-time computable* ***and****-function (both of them in the sense of Definition 1).*

Throughout the paper, we use $\mathbf{or}_{\mathrm{GI}}$ to denote the **or**-function and use $\mathbf{and}_{\mathrm{GI}}$ to denote the **and**-function of GI mentioned in Proposition 1. The existence of $\mathbf{or}_{\mathrm{GI}}$ implies that if a set $L$ disjunctive truth-table reduces to GI, then $L$ polynomial-time many-one reduces to GI. We will use this property to obtain polynomial-time many-one reductions from certain sets to GI.

### 2.3 Computational Problems on the Reconstruction of Graphs

Kelly [Kel57] first proposed the idea of generalizing the Reconstruction Conjecture to $c$-vertex-deleted subgraphs for $c > 1$. Kelly showed that there are graphs that are not determined uniquely (up to isomorphism) by their 2-vertex-deleted subgraphs. However, it is believed that, for any $c > 1$, all sufficiently large graphs satisfy the general reconstruction problem for $c$-vertex-deleted subgraphs. From a computational complexity point of view, it is interesting to analyze the complexity of problems related to the reconstruction of a graph from its $c$-vertex-deleted or $c$-edge-deleted subgraphs for different values of $c$. With this motivation, we define the computational problems we study in this paper.

1. VERTEX-DECK CHECKING$_c$ (abbreviated VDC$_c$)
   $\mathrm{VDC}_c = \{\langle G; [G_1, \ldots, G_n]\rangle \mid [G_1, \ldots, G_n] = \text{vertex-deck}_c(G)\}$.

2. EDGE-DECK CHECKING$_c$ (abbreviated EDC$_c$)
   $\mathrm{EDC}_c = \{\langle G; [G_1, \ldots, G_m]\rangle \mid [G_1, \ldots, G_m] = \text{edge-deck}_c(G)\}$.
3. LEGITIMATE VERTEX-DECK$_c$ (abbreviated LVD$_c$)
   $\mathrm{LVD}_c = \{\langle [G_1, \ldots, G_n]\rangle \mid (\exists G)[[G_1, \ldots, G_n] = \text{vertex-deck}_c(G)]\}$.
4. LEGITIMATE EDGE-DECK$_c$ (abbreviated LED$_c$)
   $\mathrm{LED}_c = \{\langle [G_1, \ldots, G_m]\rangle \mid (\exists G)[[G_1, \ldots, G_m] = \text{edge-deck}_c(G)]\}$

For any fixed $k \geq 2$, one can study the $k$-vertex-(edge-)card versions of the above-mentioned problems. These problems are denoted by $k$-VDC$_c$, $k$-EDC$_c$, $k$-LVD$_c$ and $k$-LED$_c$, respectively. We give the formal definition of $k$-VDC$_c$; the other problems are defined analogously.

1′. K-VERTEX-SUBDECK CHECKING$_c$ (abbreviated $k$-VDC$_c$)
   $k\text{-VDC}_c = \{\langle G; [G_1, \ldots, G_k]\rangle \mid [G_1, \ldots, G_k] \subseteq \text{vertex-deck}_c(G)\}$.

# 3 Reconstruction from Vertex-(Edge-)Deck

## 3.1 Reconstruction from a Complete Deck

In this section, we investigate the complexity of VDC$_c$, EDC$_c$, LVD$_c$ and LED$_c$ for any $c \geq 1$. Kratsch and Hemaspaandra [KH94] showed that GI is logspace isomorphic to VDC$_1$. By generalizing their proof, we strengthen this result, and show that, for any $c \geq 1$, GI is logspace isomorphic to VDC$_c$ as well as to EDC$_c$. Theorem 1(2) will be used in showing that for each $c \geq 1$, LED$_c$ $\leq_m^p$ GI.

**Theorem 1.** *(1) For all $c \geq 1$,* GI *is logspace isomorphic to* VDC$_c$*. (2) For all $c \geq 1$,* GI *is logspace isomorphic to* EDC$_c$*.*

Kratsch and Hemaspaandra [KH94] showed that GI $\leq_m^l$ LVD$_1$. We strengthen this result and show that, for any $c \geq 1$, GI $\leq_m^l$ LVD$_c$. Mansfield [Man82] showed that GI is polynomial-time Turing equivalent to LED$_1$. We strengthen this result and show that, for any $c \geq 1$, GI is polynomial-time isomorphic to LED$_c$.

**Theorem 2.** *(1) For every $c \geq 1$,* GI $\leq_m^l$ LVD$_c$*. (2) For every $c \geq 1$,* GI *is polynomial-time isomorphic to* LED$_c$*.*

## 3.2 Reconstruction from a Subdeck

We next investigate the complexity of problems related to the reconstruction of a graph from its partial (incomplete) deck of vertex-deleted or edge-deleted subgraphs. Theorem 3 states that GI is polynomial-time isomorphic to the $k$-card versions of VDC$_c$ and EDC$_c$ for each $c \geq 1$ and $k \geq 2$.

**Theorem 3.** *For every $c \geq 1$ and $k \geq 2$,* GI *is polynomial-time isomorphic to* $k$-VDC$_c$ *and* $k$-EDC$_c$*.*

We now consider the relative complexity of GI and $k$-LVD$_c$, and that of GI and $k$-LED$_c$, for $k \geq 2$. Lemma 1 gives an alternate characterization of an instance of 2-LVD$_c$ in terms of polynomially many instances of GI. As an immediate consequence of Lemma 1, we have that 2-LVD$_c \leq^p_{dtt}$ GI. From the explanation given in Section 2.2, it follows that 2-LVD$_c \leq^p_m$ GI.

**Lemma 1.** *For each $c \geq 1$, $[G_1, G_2]$ is a legitimate $c$-vertex-subdeck if and only if there exist $U_1 \subseteq V(G_1)$ and $U_2 \subseteq V(G_2)$, where $1 \leq ||U_1|| = ||U_2|| \leq c$, such that $G_1 - U_1$ is isomorphic to $G_2 - U_2$.*

**Proof** Fix a $c \geq 1$. Suppose that $[G_1, G_2]$ is a legitimate $c$-vertex-subdeck. By definition, there exist a graph $G$, distinct sets $T_1, T_2 \subseteq V(G)$, where $||T_1|| = ||T_2|| = c$, and isomorphisms $\psi_1$ from $G - T_1$ to $G_1$ and $\psi_2$ from $G - T_2$ to $G_2$. Clearly, $G_1 - \psi_1(T_2 - T_1)$ is isomorphic to $G - (T_1 \cup T_2)$ and $G_2 - \psi_2(T_1 - T_2)$ is isomorphic to $G - (T_1 \cup T_2)$. Thus, $G_1 - \psi_1(T_2 - T_1)$ is isomorphic to $G_2 - \psi_2(T_1 - T_2)$.

Now assume that there exist $U_1 = \{u_{1,1}, u_{1,2}, \ldots, u_{1,\ell}\} \subseteq V(G_1)$ and $U_2 = \{u_{2,1}, u_{2,2}, \ldots, u_{2,\ell}\} \subseteq V(G_2)$, where $1 \leq \ell \leq c$, such that $G_1 - U_1$ is isomorphic to $G_2 - U_2$ via $\psi$. We now construct a graph $\mathcal{G}_2$ by adding new vertices $v_{2,1}, \ldots, v_{2,c}$ in $G_2$ and by including new edges incident on them. The graph $\mathcal{G}_2$ is defined as follows. Initially, $\mathcal{G}_2 := G_2$. For each $1 \leq i \leq \ell$, add a vertex $v_{2,i}$ in $\mathcal{G}_2$ and connect $v_{2,i}$ to every vertex in $\psi(N_{G_1}(u_{1,i}) - U_1)$. For each $1 \leq i < j \leq \ell$, add an edge $\{v_{2,i}, v_{2,j}\}$ in $\mathcal{G}_2$ if and only if $\{u_{1,i}, u_{1,j}\} \in E(G_1)$. Finally, for each $1 \leq i \leq c - \ell$, add a new vertex $v_{2,\ell+i}$ and connect it to every other vertex in $\mathcal{G}_2$. We construct another graph $\mathcal{G}_1$ in a similar way. Initially, $\mathcal{G}_1 := G_1$. For each $1 \leq i \leq \ell$, add a vertex $v_{1,i}$ in $\mathcal{G}_1$ and connect $v_{1,i}$ to every vertex in $\psi^{-1}(N_{G_2}(u_{2,i}) - U_2)$. For each $1 \leq i < j \leq \ell$, add an edge $\{v_{1,i}, v_{1,j}\}$ in $\mathcal{G}_1$ if and only if $\{u_{2,i}, u_{2,j}\} \in E(G_2)$. Finally, for each $1 \leq i \leq c - \ell$, add a new vertex $v_{1,\ell+i}$ and connect it to every other vertex in $\mathcal{G}_1$.

Let $\psi' : V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ be defined as follows: $\psi'(V(G_1 - U_1)) = \psi(V(G_1 - U_1))$, for every $1 \leq i \leq \ell$, $\psi'(u_{1,i}) = v_{2,i}$ and $\psi'(v_{1,i}) = u_{2,i}$, and for every $\ell + 1 \leq i \leq c$, $\psi'(v_{1,j}) = v_{2,j}$. It can be verified that $\psi'$ is an isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$. Since $G_1$ $(= \mathcal{G}_1 - \{v_{1,1}, \ldots, v_{1,c}\})$ is a $c$-vertex-card of $\mathcal{G}_1$ and $G_2$ $(= \mathcal{G}_2 - \{v_{2,1}, \ldots, v_{2,c}\})$ is a $c$-vertex-card of $\mathcal{G}_2$, and since $\{v_{2,1}, \ldots, v_{2,c}\} \neq \psi'(\{v_{1,1}, \ldots, v_{1,c}\})$, it follows that $[G_1, G_2]$ is a legitimate $c$-vertex-subdeck. ∎

**Corollary 1.** *For every $c \geq 1$,* 2-LVD$_c \leq^p_m$ GI.

In Theorem 4(1) and Theorem 5, we obtain the polynomial-time isomorphism from GI to 2-LVD$_c$ for each $c \geq 1$, and from GI to $k$-LED$_c$ for each $c \geq 1$ and $k \geq 2$.

**Theorem 4.** *(1) For every $c \geq 1$,* GI *is polynomial-time isomorphic to* 2-LVD$_c$. *(2) For every $c \geq 1$ and $k \geq 2$,* GI $\leq^l_m$ $k$-LVD$_c$.

**Theorem 5.** *For every $c \geq 1$ and $k \geq 2$,* GI *is polynomial-time isomorphic to* $k$-LED$_c$.

## 4 Reconstruction Number of Undirected Graphs

**Definition 2 ([HP85,Myr89]).** *The ally-reconstruction number of a graph $G$ is the minimum number of one-vertex-deleted subgraphs (or 1-vertex-cards) that identify $G$ (up to isomorphism).*

Since the ally-reconstruction number of a graph $G$ is characterized by the *existence* of the same number of 1-vertex-cards of $G$, we will denote this number for $G$ by $vrn_\exists(G)$. Likewise, we use $ern_\exists(G)$ to denote the minimum number of 1-edge-cards that identify $G$. We also define an analogous definition of reconstruction number for a graph $G$, denoted by $vrn_\forall(G)$ (respectively, $ern_\forall(G)$), in which a certain number of 1-vertex-cards (respectively, 1-edge-cards), irrespective of their choice, suffice to recognize $G$. Thus, no matter which 1-vertex-cards (respectively, 1-edge-cards) an adversary selects for a graph $G$, $vrn_\forall(G)$ (respectively, $ern_\forall(G)$) many 1-vertex-cards (respectively, 1-edge-cards) are enough to identify $G$ up to isomorphism. If such a number doesn't exist, we define it to be $\infty$.

It is clear that for any graph $G$ for which $vrn_\exists(G) < \infty$ (respectively, $ern_\exists(G) < \infty$), $vrn_\exists(G) \leq vrn_\forall(G) \leq ||V(G)||$ (respectively, $ern_\exists(G) \leq ern_\forall(G) \leq ||E(G)||$). Note that $vrn_\exists(G)$ is finite for every graph $G$ if and only if the Reconstruction Conjecture is true, and $ern_\exists(G)$ is finite for every graph $G$ if and only if the Edge-Reconstruction Conjecture is true. Theorem 6 says that for any disconnected graph $G$, $vrn_\exists(G)$ (consequently, $vrn_\forall(G)$) is finite.

**Theorem 6 ([Myr89]).** *If $G$ is a disconnected graph with not all components isomorphic then $vrn_\exists(G) = 3$. Moreover, if $G$ is a disconnected graph with all components isomorphic then $vrn_\exists(G) \leq c + 2$ where $c$ is the number of vertices in a component.*

In the next lemma, we give an example of a family of disconnected graphs $G$ (parameterized by $n$, the number of vertices of the $n$'th graph in the family) for which $vrn_\exists(G) < vrn_\forall(G)$.

**Lemma 2.** *For all $n \geq 4$, there is a disconnected graph $G_n$ such that $||V(G_n)|| = n$ and $vrn_\exists(G_n) < vrn_\forall(G_n)$.*

**Proof** Let $n \geq 4$. Define the ordered pair

$$(G_n, H_n) = \begin{cases} (K_{\frac{n}{2}+1} \cup K_{\frac{n}{2}-1}, 2K_{\frac{n}{2}}) & \text{if } n \text{ is even,} \\ (K_{\frac{n-1}{2}+1} \cup K_{\frac{n-1}{2}-1} \cup K_1, 2K_{\frac{n-1}{2}} \cup K_1) & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 6, $vrn_\exists(G_n) = 3$. It is clear that $G_n$ and $H_n$ are nonisomorphic graphs. For even $n$, both $G_n$ and $H_n$ have $\frac{n}{2} + 1$ 1-vertex-cards that are isomorphic to $K_{\frac{n}{2}} \cup K_{\frac{n}{2}-1}$, and for odd $n$, both $G_n$ and $H_n$ have $\frac{n-1}{2} + 1$ 1-vertex-cards that are isomorphic to $K_{\frac{n-1}{2}} \cup K_{\frac{n-1}{2}-1} \cup K_1$. Thus, for even $n$, $vrn_\forall(G_n) \geq \frac{n}{2} + 2 > vrn_\exists(G_n) = 3$, and for odd $n$, $vrn_\forall(G_n) \geq \frac{n-1}{2} + 2 > vrn_\exists(G_n) = 3$. ∎

The Reconstruction Conjecture can be restated as follows: For each $n \geq 3$, given any collection $\mathcal{D}$ of $n$ graphs with $n-1$ vertices in each, there can be at most one 1-vertex-preimage of $\mathcal{D}$. What can we say about the number of nonisomorphic 1-vertex-preimages of a collection $\mathcal{D}$ of graphs with $n-1$ vertices in each where the size of $\mathcal{D}$ is smaller than $n$? Is there a subset of vertex-deck$_1(G)$ that suffices to identify $G$ up to isomorphism? Myrvold [Myr90] showed that for any tree $T$, the number of nonisomorphic preimages of endvertex-deck$(T)$ is exactly one; the unique preimage up to isomorphism is $T$ itself. However, the following theorem by Bryant [Bry71] says that there are graphs $G$ for which the endvertex-deck$(G)$ has more than one nonisomorphic preimage.

**Theorem 7 ([Bry71]).** *For any positive integer $k$, there exist nonisomorphic graphs $G$ and $H$, with $k$ endvertices in each, such that* endvertex-deck$(G)$ = endvertex-deck$(H)$.

Note that Theorem 7 only talks about the existence of at least two nonisomorphic 1-vertex-preimages of a certain collection consisting of $k$ 1-vertex-cards, for every $k \geq 2$. In the next theorem, we show that there is a family of multisets of $k$ graphs on $(2^{k-1}+1)n+k$ vertices with $2^n$ 1-vertex-preimages.

**Theorem 8.** *For all $k \geq 2$ and $n \geq 1$, there is a deck $\mathcal{D}$ of $k$ vertex-cards on $(2^{k-1}+1)n+k$ vertices with at least $2^n$ 1-vertex-preimages.*

**Proof** Each of the $k$ vertex-cards in $\mathcal{D}$ is identical, and defined as follows.

**1.** $x_0, \ldots, x_n$ are the vertices of the path graph $P_{n+1}$.

**2.** $y_1, \ldots, y_{k-1}$ are special selector vertices.

**3.** For $i := 1 \ldots n$,

**3.1** Let $G_i$ be the complete graph $K_{2^{k-1}}$ and let $V(G_i) = \{z_{i,I} \mid I \subseteq \{1, \ldots, k-1\}\}$.

**3.2** Connect $x_i$ to all the vertices of $G_i$.

**3.3** For each $z_{i,I} \in V(G_i)$, connect $z_{i,I}$ to each vertex $y_j$ where $j \in I$.

Consider 1-vertex-preimages $H$ of $\mathcal{D}$ of the following form.

**1.** $x_0, \ldots, x_n$ are the vertices of the path graph $P_{n+1}$.

**2.** $y_1, \ldots, y_{k-1}, y_k$ are special selector vertices.

**3.** For $i := 1 \ldots n$,

**3.1** Let $G_i$ be the complete graph $K_{2^{k-1}}$ and let $V(G_i) = \{z_{i,j} \mid j \in \{1, \ldots, 2^{k-1}\}\}$.

**3.2** Connect $x_i$ to all the vertices of $G_i$.

**3.3** The edges between the $y$-vertices and $G_i$ are defined according to one of the following two cases.

**Case 1.** Let $Y_1, \ldots, Y_{2^{k-1}}$ be an enumeration of subsets of $\{1, \ldots, k\}$ where each $||Y_j||$ is odd. For each $j \in \{1, \ldots, 2^{k-1}\}$, connect $z_{i,j}$ to each vertex $y_\ell$ where $\ell \in Y_j$.

**Case 2.** Let $Y_1, \ldots, Y_{2^{k-1}}$ be an enumeration of subsets of $\{1, \ldots, k\}$ where each $||Y_j||$ is even. For each $j \in \{1, \ldots, 2^{k-1}\}$, connect $z_{i,j}$ to each vertex $y_\ell$ where $\ell \in Y_j$.

Note that $H$ is a 1-vertex-preimage of $\mathcal{D}$, since $H - y_i$ is isomorphic to the 1-vertex-card in $\mathcal{D}$ for $1 \leq i \leq k$. As $i$ varies from 1 to $n$, in step 3.3 each time we can apply either Case 1 or Case 2. And every two distinct sequences of such choices in the construction of $H$ give rise to nonisomorphic graphs. Thus, the number of nonisomorphic 1-vertex-preimages is at least $2^n$. ∎

## 5 Open Problems

In this section, we mention some open problems. Theorem 4(1) states that, for every $c \geq 1$, GI $\equiv^p_{iso}$ 2-LVD$_c$. However, for $k > 2$ and $c \geq 1$, we do not know whether $k$-LVD$_c$ is polynomial-time equivalent to GI or is NP-complete (or neither). Since for $k > 2$ and $c \geq 1$ it is not clear, even under the assumption that the Reconstruction Conjecture is true, whether $k$-LVD$_c$ is low for PP, it is at least possible that $k$-LVD$_c$ is NP-complete.

It is also interesting to investigate the complexity of problems related to the reconstruction numbers. For instance, we define the following list of problems:

**a)** EXIST-VRN= $\{\langle G, k\rangle \mid vrn_\exists(G) \leq k\}$. **b)** UNIV-VRN= $\{\langle G, k\rangle \mid vrn_\forall(G) \leq k\}$.

**c)** EXIST-ERN= $\{\langle G, k\rangle \mid ern_\exists(G) \leq k\}$. **d)** UNIV-ERN= $\{\langle G, k\rangle \mid ern_\forall(G) \leq k\}$.

It is easy to see that EXIST-VRN $\in \Sigma_2^p$ (since GI is low for $\Sigma_2^p$), UNIV-VRN $\in$ coNP$^{\mathrm{GI}}$, EXIST-ERN $\in$ NP$^{\mathrm{GI}}$, and UNIV-ERN $\in$ coNP$^{\mathrm{GI}}$. It would be interesting to obtain tight (or tighter) bounds on the complexity of these problems. (For instance, is EXIST-VRN complete for $\Sigma_2^p$? Is UNIV-ERN coNP-hard?)

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