# Logic

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# Reasoning With Logic

## What is it?

A method to make **CORRECT** conclusions from facts

## 1. PROPOSITIONAL LOGIC

Deals with statements that are made about specific objects

John is sick          Mary likes Computer Science

If Mary likes John then Mary is crazy

## 1. PREDICATE LOGIC

Deals with statements that are made about a class of objects

All students are smart

Mary likes everything that is expensive

Everyone who works hard is successful

# Propositional Logic

Uses Propositions, Logical Connectives, and Rules of Inference

## PROPOSITIONS

A statement (or assertion) that is either TRUE or FALSE

**Simple** Propositions

John is sick    Mary likes Computer Science

**Compound** Propositions

**If** Mary likes John **then** Mary is crazy

one proposition    another proposition

The following are not propositions!   Why?

Is Mary crazy?    Is John sick?

Forget about it

# Logical Connectives

More commonly called **Logical Operators**

Used to create propositions from simpler propositions

**If** Mary likes John **then** Mary is crazy

Robert attends USF **and** Rose attends UF

**If** the outlet valve is closed **then** if the gas temperature is greater than 200 **or** the gas pressure is greater than 50 **then** safety factor is low

Represent propositions with lower case letters

**if** p **then** q

m **and** t

**if** x **then if** y **or** z **then** w

Why substitute a proposition with a symbol?

# Logical Operators

1. **NOT**

symbol: ¬ other symbols used: ~

examples:

¬ (Mary likes John)

¬ p

¬ p is called a **negation**

¬ p is also called **the negation of p**

2. **AND**

symbol: $\wedge$

examples

Robert attends USF $\wedge$ Rose attends UF

m $\wedge$ t

m $\wedge$ t is called a **conjunction**

m $\wedge$ t is also called **the conjunction of m and t**

3. **OR**

symbol: $\vee$

examples:

(gas temperature is greater than 200) $\vee$
(the gas pressure is greater than 50)

$y \vee z$

$y \vee z$ is called a **disjunction**

$y \vee z$ is also called **the disjunction of y and z**

4. **IMPLICATION (CONDITIONAL)**

symbol: $\rightarrow$

examples

Mary likes John $\rightarrow$ Mary is crazy

$p \rightarrow q$

$p \rightarrow q$ is called an **implication**

# Truth Tables

They **define** the truth values of compound propositions

### 1. **NOT**

| p | ¬ p |
|---|---|
| T | F |
| F | T |

### 2. **AND**

| p | m | p ∧ m |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

### 3. **OR**

| p | m | p ∨ m |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

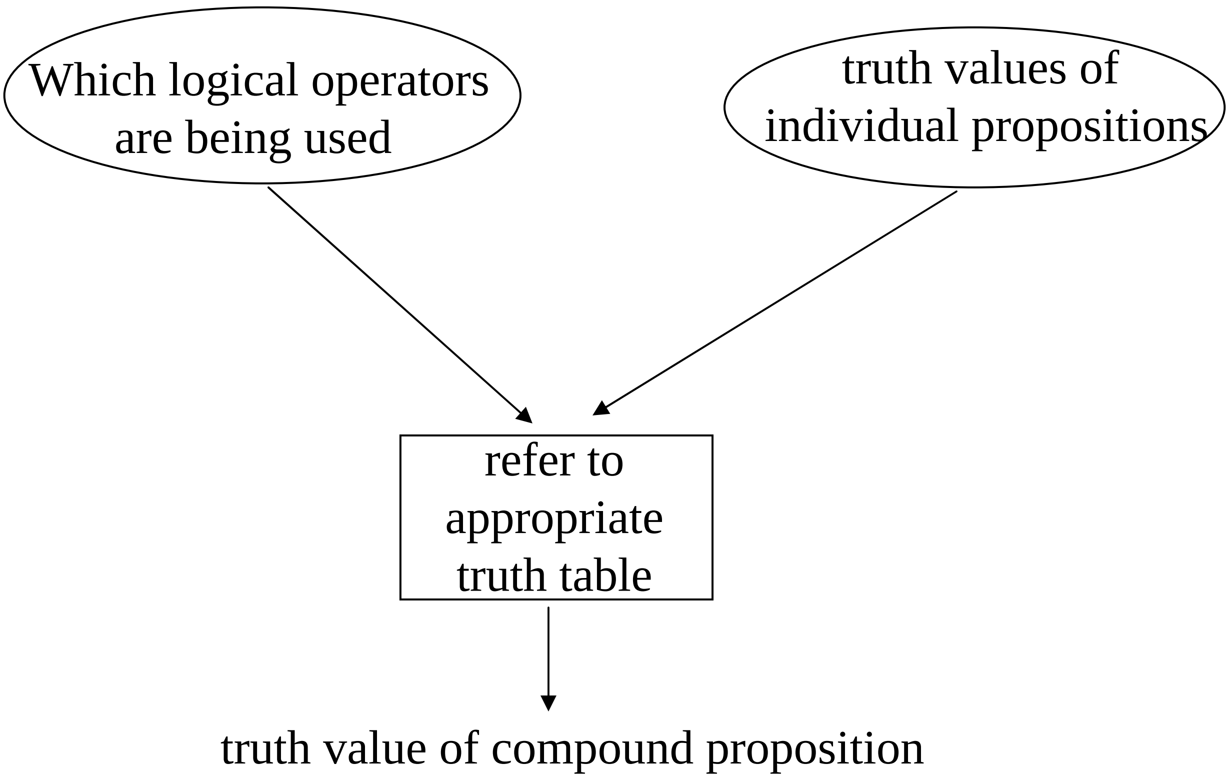
### 4. **EXCLUSIVE-OR**

| p | m | p ⊕ m |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

### 4. **IMPLICATION**

| p | m | p → m |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Truth Values of Compound Propositions




**Example**: What is the truth value of the proposition $(p \vee m) \wedge (p \rightarrow x)$ if $p = T$, $m = F$, $x = F$

Solution: the truth value of $(p \vee m)$ is: **$T \vee F = T$**

the truth value of $(p \rightarrow x)$ is: **$T \rightarrow F = F$**

the truth value of $(p \vee m) \wedge (p \rightarrow x)$ is: **$T \wedge F = F$**

**Can you think of a general method for problems like this one?**

# Truth Table of a Compound Proposition

It defines the truth value of the compound proposition for all possible combinations of truth values of the individual propositions

What is the truth **table** of the proposition $(p \vee m) \wedge (p \rightarrow x)$ seen in previous page?

| p | m | x | $p \vee m$ | $p \rightarrow x$ | $(p \vee m) \wedge (p \rightarrow x)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | T | T | T |
| F | T | F | T | T | T |
| F | F | T | F | T | F |
| F | F | F | F | T | F |

This truth table has 8 rows, that is $2^3$

How many rows will truth tables have in general ? ______

**Develop a general method to build truth tables**

Build a truth table for the proposition: $Q = [p \rightarrow (m \vee y)] \wedge (\neg m)$

## Build a Truth Table

| p | m | y | $m \vee y$ | $p \rightarrow (m \vee y)$ | $\neg m$ | $[p \rightarrow (m \vee y)] \wedge (\neg m)$ |
|---|---|---|---|---|---|---|
| T | T | T | | | | |
| T | T | F | | | | |
| T | F | T | | | | |
| T | F | F | | | | |
| F | T | T | | | | |
| F | T | F | | | | |
| F | F | T | | | | |
| F | F | F | | | | |

The compound proposition $[p \rightarrow (m \vee y)] \wedge (\neg m)$ is called a **CONTINGENCY** because sometimes its truth value is true and sometimes it is false ¬ depending on the truth values of the individual propositions

# Truth Table and Equivalence

Two (or more) propositions are equivalent if they have the same truth value for each combination of their individual propositions

To test for equivalence, build the truth table for each proposition and compare the truth value of each proposition row by row.

Build a truth table for the proposition “$m \rightarrow y$” to determine whether “$m \rightarrow y$” is equivalent to the proposition “$[p \rightarrow (m \vee y)] \wedge (\neg m)$” seen in previous page. Use the truth table already built and just add an extra column for “$m \rightarrow y$”.

Are the following propositions equivalent?

1. “$p \rightarrow m$” and “$\neg m \rightarrow \neg p$”

2. “$p \rightarrow m$” and “$\neg p \rightarrow \neg m$”

# Tautology and Contradiction

A **TAUTOLOGY** is a compound proposition that is always true regardless of the truth value of the individual propositions

Is $p \vee \neg p$ a tautology ?

We write $P \Leftrightarrow Q$ to mean that $P \leftrightarrow Q$ is a tautology. We would also say in this case that P is equivalent to Q.

How about $\neg ( p \wedge m ) \Leftrightarrow ( \neg p \vee \neg m )$ ? ( DeMorgan's law)

A **CONTRADICTION** is a compound proposition that is always false regardless of the truth value of the individual propositions

Is $p \wedge \neg p$ a contradiction ?

How about $m \leftrightarrow \neg m$

# More on Implications

$p \rightarrow m$

p is called the **premise** or **antecedent**
m is called the **conclusion** or **consequent**

**p is a sufficient condition for m**

***if it is raining then the ground is wet***

For the ground to be wet it is sufficient that it is raining but not necessary since I could wet the ground with a water hose.

p is not the only cause of m. m can be true even if p is false.

**m is a necessary condition for p**

***if it is raining then the ground is wet***

If it is raining then the ground has got to be wet

# Convert English to Logical Propositions

First use a symbol to represent each simple proposition.

Then write the compound proposition

1. If it is warm tomorrow and the wind is calm, I will go boating

p q r

$$(p \wedge q) \rightarrow r$$

2. It is necessary to work many problems to understand Discrete Math

p q

$$q \rightarrow p$$

3. To get an A in Discrete Math it is sufficient to have a class average of 90 or above

p q

$$q \rightarrow p$$

4. Either it is hot or it is humid (or both)

p q

$$p \vee q$$

5. Either it is hot or it is cold ( but not both)

p q

$$p \oplus q$$

# More on Implications

The **converse** of $p \rightarrow m$ is $m \rightarrow p$

Show whether $p \rightarrow m$ is equivalent to $m \rightarrow p$

The **contrapositive** of $p \rightarrow m$ is $\neg m \rightarrow \neg p$

Show whether $p \rightarrow m$ is equivalent to $\neg m \rightarrow \neg p$

The **inverse** of $p \rightarrow m$ is $\neg p \rightarrow \neg m$

Show whether $p \rightarrow m$ is equivalent to $\neg p \rightarrow \neg m$

# The Biconditional

symbol used $\leftrightarrow$ example $p \leftrightarrow m$

it means: $( p \rightarrow m ) \wedge ( m \rightarrow p )$

It is read as "p if and only if m"

Its truth table is

**BICONDITIONAL**

| p | m | $p \leftrightarrow m$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Laws of Algebra for Propositions

## *Idempotent Laws*

1. $p \lor p \Leftrightarrow p$

2. $p \land p \Leftrightarrow p$

## *Associative Laws*

3. $(p \lor m) \lor x \Leftrightarrow p \lor (m \lor x)$

4. $(p \land m) \land x \Leftrightarrow p \land (m \land x)$

## *Commutative Laws*

5. $p \lor m \Leftrightarrow m \lor p$

6. $p \land m \Leftrightarrow m \land p$

## *Distributive Laws*

7. $(p \lor (m \land x)) \Leftrightarrow (p \lor m) \land (p \lor x)$

8. $(p \land (m \lor x)) \Leftrightarrow (p \land m) \lor (p \land x)$

## *Identity Laws*

9. $p \lor F \Leftrightarrow p$

10. $p \land T \Leftrightarrow p$

11. $p \lor T \Leftrightarrow T$

12. $p \land F \Leftrightarrow F$

## *Complement Laws*

13. $\neg T \Leftrightarrow F$

14. $\neg F \Leftrightarrow T$

15. $\neg \neg p \Leftrightarrow p$

16. $p \lor \neg p \Leftrightarrow T$

17. $p \land \neg p \Leftrightarrow F$

## *De Morgan's Laws*

18. $\neg (p \lor m) \Leftrightarrow \neg p \land \neg m$

19. $\neg (p \land m) \Leftrightarrow \neg p \lor \neg m$

# Simplifying Compound propositions

The compound proposition $(p \to m) \wedge (\neg p \to m)$
can be simplified to the simple proposition $m$
as follows:

| | |
|---|---|
| $(p \to m) \wedge (\neg p \to m)$ | |
| $(\neg p \vee m) \wedge (\neg p \to m)$ | using $(p \to m) \Leftrightarrow (\neg p \vee m)$ |
| $(\neg p \vee m) \wedge (\neg(\neg p) \vee m)$ | using $(p \to m) \Leftrightarrow (\neg p \vee m)$ |
| $(\neg p \vee m) \wedge (p \vee m)$ | using complement law |
| $(m \vee \neg p) \wedge (p \vee m)$ | using commutative law |
| $(m \vee \neg p) \wedge (m \vee p)$ | using commutative law |
| $m \vee (\neg p \wedge p)$ | using distributive law |
| $m \vee F$ | using complement law |
| $m$ | using identity law |

Notice:

1. every step taken is justified using an algebra law or a specific previously proven well-recognized equivalence

2. Only a small part of the overall proposition is changed with every step

## Making Conclusions

Given a group of propositions ***that are true***

What conclusions can be made from them?

Example:

Assume that the following propositions are true:

1. If the suspect committed the murder then his hands will be sweating when he is interrogated

2. The suspect's hands were sweating when he was being interrogated

Conclusion: *The suspect committed the murder*

Is this a valid conclusion ?

1. $p \rightarrow m$
2. m

Conclusion: *p* Not Valid!!

**IMPLICATION**

| | p | m | $p \rightarrow m$ |
|---|---|---|---|
| ✓ | T | T | T |
| | T | F | F |
| ✓ | F | T | T |
| | F | F | T |

**This is the fallacy of affirming the consequent**

# Making Conclusions

Example:

Assume that the following propositions are true:

1. If the suspect's clothes were covered with blood when he was arrested then he committed the murder

2. The suspect's clothes were not covered with blood when he was arrested

Conclusion: *The suspect did not commit the murder*

Is this a valid conclusion ?

1. $p \rightarrow m$
2. $\neg p$

Conclusion: $\neg m$ Not Valid!!

**IMPLICATION**

| | p | m | $p \rightarrow m$ |
|---|---|---|---|
| | T | T | T |
| | T | F | F |
| ✓ | F | T | T |
| ✓ | F | F | T |

**This is the fallacy of denying the antecedent**

# Basic Rules of Inference

Inference rules allows us to make valid conclusions from a group of propositions ***that are true***

## MODUS PONENS

1. $p \rightarrow m$
2. $p$

Conclusion: *m*

### IMPLICATION

| | p | m | $p \rightarrow m$ |
|---|---|---|---|
| ✓ | T | T | T |
| | T | F | F |
| | F | T | T |
| | F | F | T |

## MODUS TOLLENS

1. $p \rightarrow m$
2. $\neg m$

Conclusion: $\neg p$

### IMPLICATION

| | p | m | $p \rightarrow m$ |
|---|---|---|---|
| | T | T | T |
| | T | F | F |
| | F | T | T |
| ✓ | F | F | T |

What valid conclusion can be made if the suspect's hands were not sweating when he was interrogated ?

# The Need for a More General Method

An Expert System Example:

1. If there are signs of throat infection and there is evidence that the organism is streptococcus then the patient has strep throat

2. If the patient's throat is red then there are signs of throat infection

3. If the stain of the organism is grampos and the morphology of the organism is coccus and the growth of the organism is chains then there is evidence that the organism is streptococcus

4. etc, etc,……...

When there are **many** propositions that are true, using the truth table method of previous pages is not feasible and using modus ponens and modus tollens is cumbersome.

We need something better !!

# Resolution

Resolution is an inference rule that can be applied to **TWO** propositions that are in disjunctive form to create a new proposition

Examples:

1. $\neg b \vee c$
2. $b \vee t$
3. $\neg p \vee m$
4. $\neg m$

Apply resolution to each pair

5. $c \vee t$
6. $\neg p$

***There must be a simple proposition that is positive in one of the pair and negative in the other. That simple proposition is eliminated and the rest are combined in one resulting disjunctive expression***

It is **INCORRECT** to do the following:
given

1. $a \vee b \vee c$
2. $\neg c \vee \neg b \vee m$

use resolution to conclude

3. $a \vee m$

***You can only eliminate one proposition at a time when using resolution!!***

# Resolution and Modus Ponens

Will Resolution arrive at the same conclusion as Modus Ponens?

1. $p \rightarrow m$
2. $p$

<u>Modus Ponens</u>

conclude: m

<u>Resolution</u>

1. $\neg p \vee m$
2. $p$

3. $m$

Do the same for Modus Tollens!!

# Converting to Disjunctions

| These are disjunctions | These are NOT disjunctions |
|---|---|
| $a \vee \neg b \vee c$ | $(a \vee \neg b) \wedge c$ |
| $m$ | $m \wedge t$ |
| $a \vee ( \neg b \vee \neg c )$ | $a \vee \neg ( b \vee c )$ |

*A compound proposition that is not a disjunction can be converted to a set of disjunctions as follows*

Case 1: If you have a conjunction of disjunctions then write each disjunction as a separate statement

Examples:

$m \wedge t$

1. $m$
2. $t$

$(a \vee b) \wedge (c \vee d) \wedge ( \neg e \vee f)$

1. $(a \vee b)$
2. $(c \vee d)$
3. $( \neg e \vee f)$

## Converting to Disjunctions

Case 1: If you have a disjunction of conjunctions then apply the distributive law and other algebra laws until you get a conjunction of disjunctions

Example:

$(a \wedge b) \vee (c \wedge \neg b)$

$[(a \wedge b) \vee c] \wedge [(a \wedge b) \vee \neg b]$ using distributive law

$[(a \vee c) \wedge (b \vee c)] \wedge [(a \wedge b) \vee \neg b]$ using distributive law

$[(a \vee c) \wedge (b \vee c)] \wedge [(a \vee \neg b) \wedge (b \vee \neg b)]$ using distributive law

$(a \vee c) \wedge (b \vee c) \wedge (a \vee \neg b) \wedge (b \vee \neg b)$ using associative law

$(a \vee c) \wedge (b \vee c) \wedge (a \vee \neg b) \wedge T$ using complement law

$(a \vee c) \wedge (b \vee c) \wedge (a \vee \neg b)$ using identity law

This is now Case 1 ¬ so separate each disjunction

1. $(a \vee c)$
2. $(b \vee c)$
3. $(a \vee \neg b)$

## Using Resolution

Assume that the following propositions are true:

1. If it is summer then it is hot
2. It is raining
3. It it is not summer then school is open
4. It is not hot or it is not raining

What conclusions can be made from these propositions?

Step 1: Convert propositions from English to symbols

s = it is summer  h = it is hot

r = it is raining  o = school is open

1. $s \rightarrow h$  2. $r$

3. $\neg s \rightarrow o$  4. $\neg h \vee \neg r$

Step 2: Convert propositions to disjunctions

1. $\neg s \vee h$  2. $r$

3. $s \vee o$  4. $\neg h \vee \neg r$

Step 3: Use resolution

1 and 3: 5. $h \vee o$

2 and 4: 6. $\neg h$

1 and 6: 7. $\neg s$

5 and 6: 8. $o$

Step 4: Convert from symbols to English

1. It is not hot
2. It is not summer
3. School is open

## Using Resolution and Contradiction

Assume that the following propositions are true:

1. $\neg p$
2. $\neg q \vee r$
3. $(q \vee m) \wedge (\neg m \vee p)$

Can it be concluded that "r" is true from these propositions?

Step 1:Assume that "$\neg r$" (the negation of the desired conclusion) is true

4. $\neg r$

Step 2: Convert propositions to disjunctions

1. $\neg p$ 2. $\neg q \vee r$

3a. $q \vee m$ 3b. $\neg m \vee p$

4. $\neg r$

Step 3: Use resolution and look for a contradiction

2 and 4: 5. $\neg q$

5 and 3a: 6. m

1 and 3b: 7. $\neg m$

6 and 7: 8. NIL means CONTRADICTION

*Since the original propositions 1, 2, and 3 are assumed to be true there can be no contradiction among them. Therefore the only cause of a contradiction has to be statement 4 ( $\neg r$) that we added. Since it is a logical impossibility to derive a contradiction from a set of true statements then $\neg r$ cannot be true. If $\neg r$ cannot be true then the only other truth value that it can have is false.* ***If $\neg r$ is false then r is true.***

# Using Resolution and Contradiction

Assume that the following propositions are true:

1. If it is June, July, or August then it is humid
2. If it is humid and hot then it is uncomfortable
3. If it is uncomfortable or I am driving then the AC is on
4. It is July or August

Can it be concluded from the above propositions that the proposition
"If it is hot then the AC is on" is true ?

Step 1: Convert propositions from English to symbols

jun = it is June    jul = it is July    aug = it is August
hum = it is humid    hot = it is hot    ac = AC is on
uncomfy = it is uncomfortable    driv = I am driving

1. $(jun \lor jul \lor aug) \rightarrow hum$    2. $(hum \land hot) \rightarrow uncomfy$
3. $(uncomfy \lor driv) \rightarrow ac$    4. $jul \lor aug$

Step 2: Convert the desired conclusion from English to symbols and negate it

5. $\neg (hot \rightarrow ac)$

Step 3: Convert propositions to disjunctions

1. $\neg (jun \lor jul \lor aug) \lor hum$
1. $( \neg jun \land \neg jul \land \neg aug) \lor hum$

1a. $\neg jun \lor hum$    1b. $\neg jul \lor hum$    1c. $\neg aug \lor hum$

## Using Resolution and Contradiction

2. $\neg$ (hum $\wedge$ hot) $\vee$ uncomfy
2. ( $\neg$ hum $\vee$ $\neg$ hot) $\vee$ uncomfy
2. $\neg$ hum $\vee$ $\neg$ hot $\vee$ uncomfy

3. $\neg$ (uncomfy $\vee$ driv) $\vee$ ac
3a. $\neg$ uncomfy $\vee$ ac 3b. $\neg$ driv $\vee$ ac

4. jul $\vee$ aug

5. $\neg$ ( $\neg$ hot $\vee$ ac)
5. hot $\wedge$ $\neg$ ac
5a. hot 5b. $\neg$ ac

Step 4: Use resolution and look for a contradiction

3a and 5b: 6. $\neg$ uncomfy
3b and 5b: 7. $\neg$ driv
2 and 5a: 8. $\neg$ hum $\vee$ uncomfy
6 and 8: 9. $\neg$ hum
1a and 9: 10. $\neg$ jun
1b and 9: 11. $\neg$ jul
1c and 9: 12. $\neg$ aug
4 and 11: 13. aug
12 and 13: 14. NIL therefore CONTRADICTION

**Therefore the proposition "If it is hot then the AC is on" is true**

# Predicate Logic

It uses the same rules as Propositional Logic but it includes

**Predicates** **Quantifiers** **Variables**

Example:

1. All cats like to chase mice
2. Garfield is a cat

You would like to conclude that: "Garfield likes to chase mice"

Propositional Logic is not able to make such conclusion !!

Predicate Logic handles these statements as follows:

$$\forall x, \forall y : \quad Cat(x) \wedge Mouse(y) \rightarrow Likes\_to\_chase(x, y)$$

$$Cat(Garfield)$$

and allows you to conclude:

$$\forall y : \quad Mouse(y) \rightarrow Likes\_to\_chase(Garfield, y)$$

## Why Quantifiers

They allow you to express facts and relationships that are true for members of a group.

*All numbers divisible by 4 are divisible by 2*

*There are no friendly alligators*

*Not all vegetarians are smart*

*Some friends of Mary are friends of Rose*

*John likes everything that is expensive*

*Isabel only likes expensive things*

*Some birds cannot fly*

# Quantifiers

## <u>Universal Quantifier</u> ∀

used to express facts and relationships that are true **FOR ALL** members of a group

Statements will have the words "for all" "for each" "for every"

It is mostly used with the IMPLICATION operator

## <u>Existential Quantifier</u> ∃

used to express facts and relationships that are true **FOR AT LEAST ONE** member of a group

Statements will have the words "for some" "there is at least one" "there exists one"

It is mostly used with the AND operator

Variant: ∃**!** means "there is exactly one" or "there exists a unique"

# Predicates

**They describe a characteristic of an object**

Garfield is a cat — Cat(Garfield)

**predicate** **object**

| | |
|---|---|
| Isabel is smart | Smart(Isabel) |
| Leo is a friendly lion | Lion(Leo) $\wedge$ Friendly(Leo) |
| Seven is a prime integer | Integer(7) $\wedge$ Prime(7) |
| John is a student | Student(John) |
| Washington is a capital city | Capital_city(Washington) |
| Washington is a capital city | Capital(Washington) $\wedge$ City(Washington) |

# Predicates

**They describe a relationship between two or more objects**

| | |
|---|---|
| Garfield and Emma are cousins | Cousins(Garfield, Emma) |

**predicate** **objects**

| | |
|---|---|
| John takes Discrete Math | Takes(John, Discrete_Math) |
| Mary takes Calculus I | Takes(Mary,Calculus_I) |
| 16 is divisible by 4 | Divisible(4, 16) |
| 4 is divisible by 2 | Divisible(2, 4) |
| Washington is the capital of USA | Capital(Washington, USA) |
| Lakeland is between Tampa and Orlando | Between(Lakeland, Tampa, Orlando) |

**The same object can be an argument to different types of predicates**

Mary is a student majoring in Computer Science

Student(Mary) $\wedge$ Majoring_in(Mary, Computer_Science) **CORRECT**

Student(Mary) $\wedge$ Majoring_in(Computer_Science) **INCORRECT**

# Using Quantifiers

*All numbers divisible by 4 are divisible by 2*

$\forall x \quad Number(x) \wedge Divisible(x, 4) \rightarrow Divisible(x, 2)$

***There are no friendly alligators***

$\neg \exists x \quad Alligator(x) \wedge Friendly(x)$

***Not all vegetarians are smart***

$\neg \forall x \quad Vegetarian(x) \rightarrow Smart(x)$

***Some friends of Mary are friends of Rose***

$\exists x \quad Friend\_of(x, Mary) \wedge Friend\_of(x, Rose)$

***John likes everything that is expensive***

$\forall x \quad Expensive(x) \rightarrow Likes(x, John)$

***Isabel only likes expensive things***

$\forall x \quad Likes(x, Isabel) \rightarrow Expensive(x)$

***Some birds cannot fly***

$\exists x \quad Bird(x) \wedge \neg fly(x)$

# Negation of Quantifiers

The negation operator interacts with quantifiers in a simple way.

When we say that **it is false that a P(x) holds for all possible values of x**, we mean **there is at least one value of x for which P(x) is false**.

Formally: $\neg \forall x \ P(x) \leftrightarrow \exists x \, (\neg P(x))$

Similarly, when we say that **it is false to say that P(x) is true for some value of x** we mean that no matter what value of x is chosen, P(x) will be false, i.e., **P(x) is false for all values of x.**

Formally: $\neg \exists x \ P(x) \leftrightarrow \forall x \, (\neg P(x))$

Applying these rules in succession we may "simplify" a negated quantified expression so that the negation symbol does not appear to the left of any quantifier:

$$\begin{aligned} \neg \exists x \, \forall y \, \exists z \, P(x,y,z) &\leftrightarrow \forall x \, \neg \forall y \, \exists z \, P(x,y,z) \\ &\leftrightarrow \forall x \, \exists y \, \neg \exists z \, P(x,y,z) \\ &\leftrightarrow \forall x \, \exists y \, \forall z \, \neg P(x,y,z) \end{aligned}$$

# Equivalence Between ∀ and ∃

Any logical statement using one quantifier can be converted to another statement using the other quantifier.

Assume a predicate PRED and a variable x.
The following are equivalent:

*true for all* *cannot find one for which it is false*

$\forall$ x [ PRED(x)] $\Leftrightarrow$ $\neg$ {$\exists$ x [ $\neg$ PRED(x) ]}

*false for all* *cannot find one for which it is true*

$\forall$ x [ $\neg$ PRED(x)] $\Leftrightarrow$ $\neg$ {$\exists$ x [ PRED(x) ]}

*not all true* *can find at least one for which it is false*

$\neg$ {$\forall$ x [ PRED(x)] } $\Leftrightarrow$ {$\exists$ x [ $\neg$ PRED(x)
]}

*not all false* *can find at least one for which it is true*

$\neg$ {$\forall$ x [ $\neg$ PRED(x)] } $\Leftrightarrow$ {$\exists$ x [ PRED(x)
]}

## Equivalent Statements

***All numbers divisible by 4 are divisible by 2***

$\forall x \quad Number(x) \land Divisible(x, 4) \rightarrow Divisible(x, 2)$

$\neg \exists x \quad Number(x) \land Divisible(x, 4) \land \neg Divisible(x, 2)$

***There are no friendly alligators***

$\neg \exists x \quad Alligator(x) \land Friendly(x)$

$\forall x \quad \neg Alligator(x) \lor \neg Friendly(x)$

$\forall x \quad Alligator(x) \rightarrow \neg Friendly(x)$

***Not all vegetarians are smart***

$\neg \forall x \quad Vegetarian(x) \rightarrow Smart(x)$

$\exists x \quad Vegetarian(x) \land \neg Smart(x)$

***Some friends of Mary are friends of Rose***

$\exists x \quad Friend\_of(x, Mary) \land Friend\_of(x, Rose)$

$\neg \forall x \quad \neg Friend\_of(x, Mary) \lor \neg Friend\_of(x, Rose)$

$\neg \forall x \quad Friend\_of(x, Mary) \rightarrow \neg Friend\_of(x, Rose)$

# Equivalent Statements

***John likes everything that is expensive***

$\forall x \quad Expensive(x) \rightarrow Likes(x, John)$

$\neg \exists x \quad Expensive(x) \wedge \neg Likes(x, John)$

***Isabel only likes expensive things***

$\forall x \quad Likes(x, Isabel) \rightarrow Expensive(x)$

$\neg \exists x \quad Likes(x, Isabel) \wedge \neg Expensive(x)$

***Some birds cannot fly***

$\exists x \quad Bird(x) \wedge \neg fly(x)$

$\neg \forall x \quad Bird(x) \rightarrow fly(x)$

If you are not yet confused --- try explaining the difference between the following two logical statements. Which of them represents correctly the statement

***"Not all prime integers are odd"***

$\neg \forall x \quad Integer(x) \wedge Prime(x) \rightarrow Odd(x)$

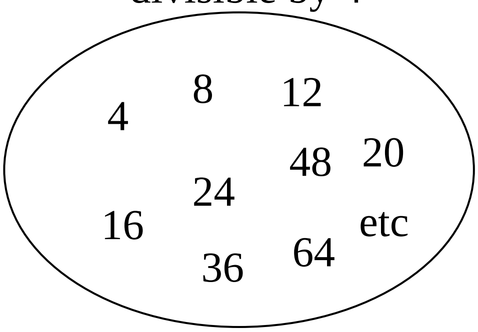
$\neg \forall x \quad Integer(x) \wedge Prime(x) \wedge Odd(x)$

## Universal Instantiation

It is an inference rule used with universal quantifiers

It means that if something is true of everything in a universe then it is true of any specific thing in that universe.

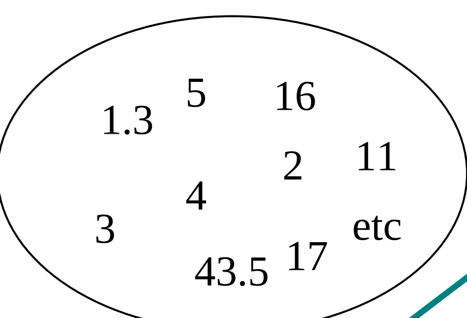
Universe of numbers divisible by 4




$\forall x \quad$ Divisible(x, 2)

Divisible(4, 2)
Divisible(8, 2)
Divisible(24,2)

Universe of numbers




$\forall x \quad$ Divisible(x, 4) $\rightarrow$ Divisible(x, 2)

Divisible(4, 4) $\rightarrow$ Divisible(4, 2)
Divisible(16, 4) $\rightarrow$ Divisible(16, 2)
Divisible(1.3, 4) $\rightarrow$ Divisible(1.3, 2)

**selects those in the universe for whom the consequent is true**

# Using Resolution

Assume that the following propositions are true:

1. All cats like to chase mice
2. Garfield is a cat

What conclusions can be made from these propositions?

Step 1: Convert propositions from English to symbols using predicates and quantifiers

$$\forall x, \forall y : \; Cat(x) \land Mouse(y) \rightarrow Likes\_to\_chase(x, y)$$
$$Cat(Garfield)$$

Step 2: Convert propositions to disjunctions dropping universal quantifiers. All variables are assumed to be universally quantified

1. $\neg Cat(x) \lor \neg Mouse(y) \lor Likes\_to\_chase(x, y)$
2. $Cat(Garfield)$

## Using Resolution

Step 3: Use Universal instantiation where appropriate. Look for pair of statements with same predicates ¬ positive in one and negative in the other and a variable argument in one and a specific object in the other

3. $\neg$ Cat(Garfield) $\vee$ $\neg$ Mouse(y) $\vee$ Likes_to_chase(Garfield, y)

Step 4: Use resolution on pairs of statements with same predicates ¬ positive in one and negative in the other, and with identical arguments within the parentheses

2 and 3: 4. $\neg$ Mouse(y) $\vee$ Likes_to_chase(Garfield, y)

4. Mouse(y) $\rightarrow$ Likes_to_chase(Garfield, y)

Step 5: Convert from symbols to English

Garfield likes to chase anything that is a mouse !!

## Using Resolution

Given the following predicate logic statements:

1. $\forall x, \forall y : Female(x) \wedge Uncle(y,x) \rightarrow Niece(x, y)$
2. Female(Joan)
3. Uncle( Peter, Joan)
4. Uncle(Peter, Michael)
5. Male(Michael)

Which conclusions can be made using resolution?

Step 2: Convert propositions to disjunctions dropping universal quantifiers.

1. $\neg Female(x) \vee \neg Uncle(y,x) \vee Niece(x, y)$
2. Female(Joan)
3. Uncle( Peter, Joan)
4. Uncle(Peter, Michael)
5. Male(Michael)

Step 3: Use Universal instantiation where appropriate. Look for pair of statements with same predicates ¬ positive in one and negative in the other and a variable argument in one and a specific object in the other

6. $\neg Female(Joan) \vee \neg Uncle(Peter, Joan) \vee Niece(Joan, Peter$

# Using Resolution

Step 4: Use resolution on pairs of statements with same predicates ¬ positive in one and negative in the other, and with identical arguments within the parentheses

2 and 6: 7. $\neg$ Uncle(Peter, Joan) $\vee$ Niece(Joan, Peter)

3 and 7: 8. Niece(Joan, Peter)